ONE STEADY-STATE HEAT-CONDUCTION PROBLEM FOR A POLYGONAL REGION WITH MIXED BOUNDARY CONDITIONS

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An exact solution is given for one heat-conduction problem involving a polygonal region with mixed boundary conditions. The solution method is based on development of a method presented by the author in previous publications.

An approach associated with conformal mapping is effective in the solution of steady-state problems with boundary conditions of the first and second type. It is quite difficult to apply this method to problems with a boundary condition of the third type, owing to the fact that a condition of the third type with constant coefficients becomes a condition with variable coefficients when conformal mapping is used. There are few studies that give an exact solution to problems with a boundary condition of the third type [1-3].

Our aim is to solve a specific heat-conduction problem with mixed boundary conditions. The exact formulation is given below. The idea behind the solution consists in the following. Conformal mapping is used to transform the problem (Fig. 1a) into the corresponding problem for a quadrant (Fig. 1b) on one side of which we have a condition of the first type, and on the other a condition of the third type with variable coefficients. The general method presented in [4] is used to construct an analytic function whose real part is a solution to this problem.

We are required to solve the steady-state heat-conduction problem for the region shown in Fig. 1a:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

with the boundary conditions

$$T|_{x=-b} = T_0, \ -\infty < y < \infty; \ T|_{x=0} = T_0, \ -\infty < y < 0; \tag{2a}$$

$$-\frac{\partial T}{\partial y} + hT|_{y=0} = 0 \ , \ 0 < x < \infty, \tag{2b*}$$

where h is the heat-transfer coefficient.

We seek a solution of the problem T that is continuous up to the boundary; we require that the functions $\partial T/\partial x$, $\partial T/\partial y$ be continuous at all points on the boundary, except for (0, 0).

We seek a solution that is bounded at infinity,

$$T(x, y) = O(1) \text{ for } r \to \infty, \tag{3}$$

where $r = \sqrt{x^2 + y^2}$.

It follows from physical considerations that

$$T(x, y) = T_0 + d \sin\left(\frac{\pi}{b} x\right) \exp\left(\frac{\pi}{b} y\right) + \exp\left(\frac{2\pi}{b} y\right) \rho(x, y)^{*)}, \quad -b \leqslant x \leqslant 0, \ y \to -\infty, \tag{4}$$

*We assume that as we move away from the boundary the ambient temperature drops rapidly to zero.

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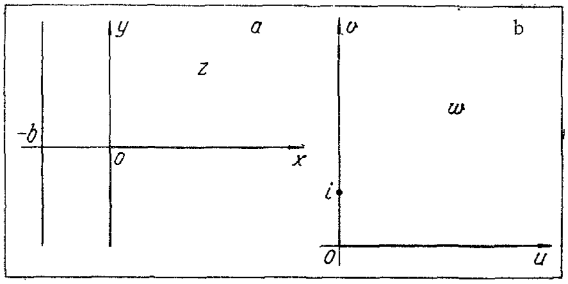

Fig. 1. Region for which the solution is sought (a), and region obtained after conformal mapping (b).

[The asymptotic behavior of the solution in this case must be the same as in the problem for a half strip whose side edges are at a constant temperature T_0]. In Eq. (4) d is a constant, while $\rho(x, y)$ is a function that is continuous and bounded together with its first-order partial derivative in the half-strip $-b \le x \le 0$, $-\infty < y \le \delta < 0$.

We shall treat the plane (x, y) as the plane of the complex variable $z = x + iy$, and conformally map our region (Fig. 1a) onto the first quadrant of the plane $w = u + iv$ (Fig. 1b). The mapping is defined by the function

$$z = \frac{2b}{\pi} \int_0^w \frac{\zeta^2}{\zeta^2 + 1} d\zeta, \tag{5}$$

where we integrate over an arbitrary curve in the region $0 < \arg w < \pi/2$.

With such a mapping, the real semiaxis $0 < x < \infty$, $y = 0$ of the z plane goes into the real semiaxis $0 < u < \infty$, $v = 0$ of the w plane. The sides $x = 0$, $-\infty < y < 0$; $x = -b$, $-\infty < y < \infty$ go, respectively, into the parts of the imaginary axis $u = 0$, $0 < v < 1$; $u = 0$, $1 < v < \infty$.

The Laplace equation is invariant under a transformation of coordinates defined by the conformal mapping. Thus the function $T = T(x(u, v), y(u, v))$ satisfies the Laplace equation

$$\frac{\partial^2 T}{\partial u^2} + \frac{\partial^2 T}{\partial v^2} = 0. \tag{6}$$

The boundary conditions for the function T have the form

$$T|_{u=0} = T_0, \quad 0 < v < \infty; \tag{7a}$$

$$-(u^2 + 1) \frac{\partial T}{\partial v} + \varepsilon u^2 T|_{v=0} = 0, \quad 0 < u < \infty, \tag{7b}$$

where we let

$$\varepsilon = \frac{2hb}{\pi}. \tag{8}$$

We note that at the singularity (0, 1) the continuity of the functions $\partial T/\partial u$, $\partial T/\partial v$ is preserved. This can be shown in the following manner.

The mapping (5) specifies the relationship between the old and new systems of curvilinear coordinates: $x = x(u, v)$, $y = y(u, v)$. Using (4), we establish the continuity of the derivatives

$$\frac{\partial T}{\partial u} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial u}; \quad \frac{\partial T}{\partial v} = \frac{\partial T}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial T}{\partial y}\frac{\partial y}{\partial v}.$$

Following the notions developed in [2, 5], we seek a solution of the problem in the form

$$T = \mathrm{Re}\Psi(w), \tag{9}$$

where $\Psi(w)$ is a function analytic in any finite portion of the region $0 < \arg w < \pi/2$ of the complex variable $w = u + iv$. In accordance with (7a), (7b) this function must satisfy the boundary conditions

$$\mathrm{Re}\,\Psi(w)|_{w=iv} = T_0, \quad 0 < v < \infty; \tag{10a}$$

$$\mathrm{Re}\left[-i(w^2 + 1)\frac{d\Psi(w)}{dw} + \varepsilon w^2 \Psi(w)\right]\Bigg|_{w=u} = 0, \quad 0 < u < \infty. \tag{10b}$$

We shall seek a solution $\Psi(w)$ continuous on the boundary, that satisfies the condition at infinity:

$$\Psi(w) = -i\frac{2}{\pi} T_0 \ln w + O(|w|^{-1}),$$
$$|w| \to \infty, \quad 0 \leqslant \arg w \leqslant \pi/2. \tag{11}$$

It is now convenient to replace $\Psi(w)$ by the new function $\chi(w)$, for which (10a) will be homogeneous:

$$\Psi(w) = \chi(w) - i\frac{2}{\pi}T_0 \ln w; \tag{12}$$

$$\operatorname{Re}\chi(w)|_{w=iv} = 0, \quad 0 < v < \infty; \tag{13a}$$

$$\operatorname{Re}\left[-i(w^2+1)\frac{d\chi(w)}{dw} + \varepsilon w^2\chi(w)\right]\Bigg|_{w=u}$$

$$= \frac{2}{\pi}T_0\frac{u^2+1}{u}, \quad 0 < u < \infty. \tag{13b}$$

Here $\chi(w)$ has the following asymptotic behavior:

$$\chi(w) = O(|\ln w|), \ |w| \to 0; \ \chi(w) = O(|w|^{-1}), \ |w| \to \infty,$$
$$0 \leqslant \arg w \leqslant \frac{\pi}{2}. \tag{14}$$

The conditions (13a), (13b), (14) for $\chi(w)$ follow from the corresponding conditions (10a), (10b), (11) for $\Psi(w)$.

The bracketed expression in (13b) is a function analytic in any finite portion of the first quadrant, and continuous at all points on the boundary except for w = 0.

Employing the method of [4], we seek it in the form

$$-i(w^2+1)\frac{d\chi(w)}{dw} + \varepsilon w^2\chi(w) = \frac{2}{\pi}T_0\frac{w^2+1}{w} + iT_0\alpha, \tag{15}$$

where α is a real constant.* It is so selected that the function $\chi(w)$, found from (15), satisfies the boundary condition (13a).

Integrating (15), we obtain

$$\chi(w) = T_0\left(\frac{w-i}{w+i}\right)^{\varepsilon/2}\exp(-i\varepsilon w)\int_w^\infty\left(\frac{\zeta+i}{\zeta-i}\right)^{\varepsilon/2}\exp(i\varepsilon\zeta)\left[-\frac{2}{\pi}\frac{i}{\zeta} + \frac{\alpha}{\zeta^2+1}\right]d\zeta, \tag{16}$$

where we make allowance for the second relationship of (14). In (16), we choose those branches of the functions $(w-i)^{\varepsilon/2}$, $(w+i)^{\varepsilon/2}$, that approach $u^{\varepsilon/2}$ along the positive real axis when $w \to \infty$. We integrate over any path in the first quadrant that does not pass through the point w = 0, w = i. In particular, for w = iv, 1 < v < ∞, we take the ray running along the boundary of the region as the path of integration. As we see, for 1 < v < ∞, the boundary condition (13a) is satisfied regardless of the value of α.

Performing certain manipulations in (16), we obtain

$$\chi(w)|_{w=iv} = -iT_0\left(\frac{v-1}{v+1}\right)^{\varepsilon/2}\exp(\varepsilon v)\int_v^\infty\left(\frac{s+1}{s-1}\right)^{\varepsilon/2}\exp(-\varepsilon s)\left[\frac{2}{\pi}\frac{1}{s} + \frac{\alpha}{s^2-1}\right]ds,$$
$$1 < v < \infty, \tag{17}$$

and thus

$$\operatorname{Re}\chi(w)|_{w=iv} = 0, \quad 1 < v < \infty. \tag{18}$$

We shall now show that by an appropriate choice of α we can satisfy the boundary condition (13a) for 0 < v < 1 as well. To do this, it is convenient to transform (16) to the form

$$\chi(w) = T_0\left(\frac{w-i}{w+i}\right)^{\varepsilon/2}\exp(-i\varepsilon w)\Bigg\{B + i\frac{2}{\pi}\Bigg[\int_0^w\Bigg(\left(\frac{\zeta+i}{\zeta-i}\right)^{\varepsilon/2}$$
$$-\exp\left(i\varepsilon\frac{\pi}{2}\right)\Bigg)\frac{\exp(i\varepsilon\zeta)}{\zeta}d\zeta - \exp\left(i\varepsilon\frac{\pi}{2}\right)\int_w^\infty\frac{\exp(i\varepsilon\zeta)}{\zeta}d\zeta\Bigg] - \alpha\int_0^w\left(\frac{\zeta+i}{\zeta-i}\right)^{\varepsilon/2}\frac{\exp(i\varepsilon\zeta)}{\zeta^2+1}d\zeta\Bigg\}, \tag{19}$$

where B is a constant:

$$B = -i\frac{2}{\pi}\int_0^\infty\left[\left(\frac{\zeta+i}{\zeta-i}\right)^{\varepsilon/2} - \exp\left(i\varepsilon\frac{\pi}{2}\right)\right]\frac{\exp(i\varepsilon\zeta)}{\zeta}d\zeta$$

*It is essential that the coefficients on $d\chi(w)/dw$, $\chi(w)$ in (15) contain only even powers of w.

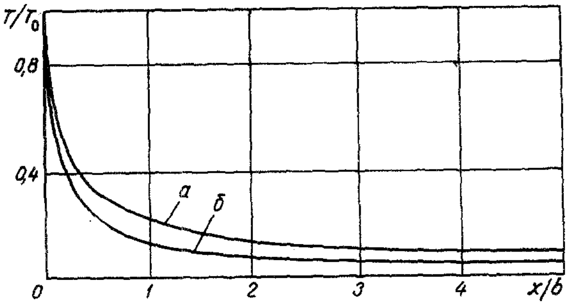

Fig. 2. Temperature distribution along radiating wall: a) $\varepsilon = 1$; b) $\varepsilon = 2$.

$$+\alpha \int\limits_0^\infty \left(\frac{\zeta+i}{\zeta-i}\right)^{\varepsilon/2} \frac{\exp(i\varepsilon\zeta)}{\zeta^2+1}\, d\zeta, \tag{20}$$

In (20) we integrate along any curve in the first quadrant that does not pass the point $\zeta = i$; here the first integral converges at the point $\zeta = 0$, since

$$\left(\frac{\zeta+i}{\zeta-i}\right)^{\varepsilon/2}\Bigg|_{\zeta=0} = \exp\left(i\varepsilon\,\frac{\pi}{2}\right).$$

In particular, we can take the real semiaxis as the path of integration. Using (19) and (20), we find that the requirement

$$\mathrm{Re}\,\chi(w)\,|_{w=iv} = 0,\ 0<v<1 \tag{21}$$

leads to a solution of the linear algebraic equation for α. In fact, if in (19) we integrate over the segment of the imaginary axis between the points $w = 0$; $w = iv$, $0 < v < 1$, and separate the real axis, we obtain

$$\mathrm{Re}\,\chi(w)\,|_{w=iv} = T_0\left(\frac{1-v}{1+v}\right)^{\varepsilon/2} \exp(\varepsilon v)\,\mathrm{Re}\left[B\exp\left(-i\varepsilon\,\frac{\pi}{2}\right)\right],\ 0<v<1. \tag{22}$$

Comparing (21) and (22), we have

$$\mathrm{Re}\left[B\exp\left(-i\varepsilon\,\frac{\pi}{2}\right)\right] = 0. \tag{23}$$

Expression (23) is a linear algebraic equation for α. Solving, we have

$$\alpha = \frac{1-\dfrac{2}{\pi}\displaystyle\int\limits_0^\infty \sin[\varepsilon(s-\mathrm{arctg}\,s)]\frac{ds}{s}}{\displaystyle\int\limits^\infty \cos[\varepsilon(s-\mathrm{arctg}\,s)]\,\frac{ds}{s^2+1}}. \tag{24}$$

The point $w = i$ is a removable singularity,

$$\chi(i) = \lim_{w\to i}\chi(w) = -iT_0\,\frac{\alpha}{\varepsilon}. \tag{25}$$

Considering (12) and (16) together, we write the solution (6)-(9),

$$\Psi(w) = T_0\left\{\left(\frac{w-i}{w+i}\right)^{\varepsilon/2}\exp(-i\varepsilon w)\int\limits_w^\infty\left(\frac{\zeta+i}{\zeta-i}\right)^{\varepsilon/2}\exp(i\varepsilon\zeta)\left[-\frac{2}{\pi}\,\frac{i}{\zeta}+\frac{\alpha}{\zeta^2+1}\right]d\zeta - i\,\frac{2}{\pi}\ln w\right\}. \tag{26}$$

In (26), the relationship between the variables z and w is given by (5), while the desired temperature T is found as the real part of the function Ψ.

The exact formulas obtained were used in numerical calculations for the temperature distribution $T = T(x, 0)$ along a radiating wall (Fig. 1a). This distribution is found from (26) and (5),

$$T = T_0\left\{\frac{2}{\pi}\int\limits_u^\infty \sin\left[\varepsilon\left(s-u-\mathrm{arctg}\frac{s-u}{su+1}\right)\right]\frac{ds}{s} + \alpha\int\limits_u^\infty \cos\left[\varepsilon\left(s-u-\mathrm{arctg}\,\frac{s-u}{su+1}\right)\right]\frac{ds}{s^2+1}\right\}, \tag{27}$$

where the relationship between x, u coordinates is given by the formula

$$x = \frac{2b}{\pi}(u-\mathrm{arctg}\,u). \tag{28}$$

For large x, the value of the temperature is represented by an asymptotic formula that follows from the exact solution,

$$T(x, 0) \approx T_0\left(\frac{2}{\pi}\right)^2\frac{1}{\varepsilon}\,\frac{b}{x},\ x\to\infty. \tag{29}$$

Calculations were carried out for the following values of ε (8): a) 1; b) 2. These parameters correspond to halving of the temperature in the ambient at the following distances from the radiating wall: a) 0.44 b; b) 0.22 b.

The calculations were formed on a Razdan-2 computer. The results are given as graphs in Fig. 2.

NOTATION

T_0 is the temperature of the side faces;
T is the temperature within the region;
h is the heat-transfer coefficient;
b is the distance between the side faces.

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